<u>Definition</u> (Euclidean distance d_E)

Let $P(x_1, y_1)$ and $Q(x_1, y_1)$ denote two points in the Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$. The Euclidean distance d_E is given by

$$d_E = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

12. Let $\mathcal{C} = \{\mathbb{R}, \mathcal{L}_E\}$ denote the Cartesian Plane and let $P(x_1, y_1)$ and $Q(x_1, y_1)$ denote two arbitrary points from Cartesian Plane. Show that Euclidean distance $d_E = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is a distance function.

<u>Definition</u> (Poincaré distance d_H)

Let $P(x_1, y_1)$ and $Q(x_1, y_1)$ denote two points in the Poincaré Plane $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$. The Poincaré distance d_H is given by

$$d_H = \begin{cases} |\ln(y_2) - \ln(y_1)|, & \text{if } x_1 = x_2 \\ |\ln(\frac{x_1 - c + r}{y_1}) - \ln(\frac{x_2 - c + r}{y_2})|, & \text{if } P, Q \in {}_cL_r \end{cases}.$$

13. Let $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ denote the Poincaré Plane and let $P(x_1, y_1)$ and $Q(x_1, y_1)$ denote two arbitrary points from Poincaré Plane. Show that Poincaré distance d_H is a distance function.

<u>Definition</u> (ruler or coordinate system)

Let ℓ be a line in an incidence geometry $\{\mathcal{S}, \mathcal{L}\}$. Assume that there is a distance function d on $\mathcal{S}$. A function $f : \ell \to \mathbb{R}$ is a ruler (or coordinate system) for ℓ if

(i) f is a bijection;

(ii) for each pair of points P and Q on ℓ

$$|f(P) - f(Q)| = d(P, Q). \tag{1}$$

Equation (1) is called the Ruler Equation and $f(P)$ is called the coordinate of P with respect to f.

14. Let $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ denote the Cartesian Plane and let d denote the Euclidean distance. Define function $f : L_{2,3} \to \mathbb{R}$ on the following way: $f(Q) = f((x, y)) = x\sqrt{5}$, $\forall Q \in L_{2,3}$. Show that f is a ruler for $L_{2,3}$ and find the coordinate of $R(1, 5)$ with respect to f.

15. Let $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ denote the Poincaré Plane and let d denote the Poincaré distance. Define function $g : {}_4L_9 \to \mathbb{R}$ on the following way: $g(P) = g((x, y)) = \ln \frac{x+5}{y}$, $\forall P \in {}_4L_9$. Show that g is a ruler for ${}_4L_9$ and find the coordinate of $M(5, 2\sqrt{3})$ with respect to g.

16. Let $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ denote the Cartesian Plane and let d denote the Taxicab distance. Define function $h : L_{-2,3} \to \mathbb{R}$ on the following way: $h(R) = h((x, y)) = 3x$, $\forall R \in L_{-2,3}$. Show that h is a ruler for $L_{-2,3}$ and find the coordinate of $N(1, 1)$ with respect to h.

<u>Definition</u> (Ruler Postulate, metric geometry)

An incidence geometry $\{\mathcal{S}, \mathcal{L}\}$ together with a distance function d satisfies the Ruler Postulate if every line $\ell \in \mathcal{L}$ has a ruler. In this case we say $\mathcal{M} = \{\mathcal{S}, \mathcal{L}, d\}$ is a metric geometry.

17. Show that the Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ with the Euclidean distance, d_E, is a metric geometry.

<u>Definition</u> (Euclidean Plane)

The Euclidean Plane is the model $\mathcal{E} = \{\mathbb{R}, \mathcal{L}_E, d_E\}$.

18. Show that the Poincaré Plane $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ with the Poincaré distance, d_H, is a metric geometry.

Convention. From now on, the terminology Poincaré Plane and the symbol $\mathcal{H}$ will include the hyperbolic distance d_H:

$$\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H, d_H\}.$$

19. Show that the Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ with the Taxicab distance, d_T, is a metric geometry.

Definition (Taxicab Plane)
The model $\mathcal{T} = \{\mathbb{R}, \mathcal{L}_E, d_T\}$ will be called the Taxicab Plane.

Lets summarizes the rulers which we have discussed for the three major models of a metric geometry.

Model	Type of line	Standard Ruler or coordinate system for line
Euclidean Plane $\mathcal{E}$	$L_a = \{(a,y) \mid y \in \mathbb{R}\}$ $L_{m,b} = \{(x,y) \in \mathbb{R}^2 \mid y = mx + b\}$	$f(a,y) = y$ $f(x,y) = x\sqrt{1+m^2}$
Poincaré Plane $\mathcal{H}$	${}_aL = \{(a,y) \in \mathbb{H} \mid y > 0\}$ ${}_cL_r = \{(x,y) \in \mathbb{H} \mid (x-c)^2 + y^2 = r^2\}$	$f(a,y) = \ln y$ $f(x,y) = \ln \frac{x-c+r}{y}$
Taxicab Plane $\mathcal{T}$	$L_a = \{(a,y) \mid y \in \mathbb{R}\}$ $L_{m,b} = \{(x,y) \in \mathbb{R}^2 \mid y = mx + b\}$	$f(a,y) = y$ $f(x,y) = (1+\lvert m\rvert)x$

Convention. In discussions about one of the three models above, the coordinate of a point with respect to a line ℓ will always mean the coordinate with respect to the standard ruler for that line as given in the above table.

In the next section we will discuss some special rulers for a line. These should not be confused with the standard rulers defined above.

20. In the Euclidean Plane $\mathcal{E} = \{\mathbb{R}^2, \mathcal{L}_E, d_E\}$, (i) find the coordinate of $M(2,3)$ with respect to the line $x = 2$; (ii) find the coordinate of $M(2,3)$ with respect to the line $y = -4x + 11$. (Note that your answers are different.)

21. Find the coordinate of $M(2,3)$ with respect to the line $y = -4x + 11$ for the Taxicab Plane $\mathcal{T} = \{\mathbb{R}^2, \mathcal{L}_E, d_T\}$. (Compare with Problem 20.)

22. Find the coordinates in $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H, d_H\}$ of $M(2,3)$ (i) with respect to the line $(x-1)^2 + y^2 = 10$; (ii) with respect to the line $x = 2$.

23. Find the Poincaré distance between
 i. $A(1,2)$ and $B(3,4)$;
 ii. $P(2,1)$ and $Q(4,3)$.

24. Find a point P on the line $L_{2,-3}$ in the Euclidean Plane whose coordinate is -2.

25. Find a point P on the line $L_{2,-3}$ in the Taxicab Plane whose coordinate is -2.

26. Find a point P on the line ${}_{-3}L_{\sqrt{7}}$ in the Poincaré Plane whose coordinate is in $\ln 2$.

27. We shall define a new distance d^* on $\mathbb{R}^2$ by using d_E. Specifically:

$$d^*(P,Q) = \begin{cases} d_E(P,Q), & \text{if } d_E(P,Q) \le 1 \\ 1, & \text{if } d_E(P,Q) > 1 \end{cases}$$

(i) Prove that d^* is a distance function. (ii) Find and sketch all points $P \in \mathbb{R}^2$ such that $d^*((0,0),P) \le 2$. (iii) Find all points $P \in \mathbb{R}^2$ such that $d^*((0,0),P) = 2$.